WHY ABSTRACT ALGEBRA FOR PRE-SERVICE PRIMARY SCHOOL TEACHERS

Eleni Agathocleous
Cyprus Ministry of Education and Culture

ABSTRACT There has been a rapid increase in the volume of research papers focusing on mathematics teacher education and in particular on the characteristics that should describe the desired mathematical knowledge for teachers. In this paper, it is suggested through theoretical discussion on the character of Abstract Algebra and via Peirce’s semiotics, that a one-semester course in Abstract Algebra, specially designed for pre-service primary school teachers, can enhance the development of the basic, multifaceted characteristic of connectedness. Some concrete examples are given in the end of the paper, taken from in-class observations and reflections on a similar course taught by the author, for two consecutive semesters.

Key words: Teachers’ mathematical knowledge, Connectedness, Abstract Algebra, Semiotics.

INTRODUCTION

In this paper I argue that a course in basic notions of Abstract Algebra, specially designed for pre-service primary school teachers, can help teachers regroup and enrich their existing mathematical knowledge, as well as achieve connectedness across various mathematical domains as well as across time, as a mathematical idea develops and extends. These basic notions of Abstract Algebra include the notions of group, ring and field, accompanied by a very basic introduction to functions, modular arithmetic and complex numbers. Unlike a standard introductory course in Abstract Algebra intended for mathematicians, the aim of this course would not be the teaching of group theory, quotient rings or maximal ideals for example. But instead, the introduction of groups, rings and fields as a means to put existing mathematical knowledge (such as the number systems N, Z, Q, R) or new mathematical knowledge (such as the number systems C and Z_n) into new forms, in order to reveal more information on the structure of these mathematical systems as well as present new ways of understanding old knowledge. The theoretical discussion in this paper focuses on the unifying character of the basic notions of group, ring and field, and I employ Peirce’s Semiotics to show how the use of these notions is on the highest level of the hierarchy, helping the learner make the desired connections.

TEACHERS’ MATHEMATICAL KNOWLEDGE

What mathematical knowledge does it take to teach primary mathematics well? Is it a matter of quantity? Of depth? Or is it a matter of different quality? There has been a rapid increase in the volume of research papers focusing on the relatively new area of mathematics teacher education (Adler, Ball, Krainer, Lin, & Novotna, 2005) and investigating the “unsolved” (Ball, Lubienski, & Mewborn, 2001) problem of teachers’ mathematical knowledge (see, e.g., Davis & Simmt, 2006; Ball & Bass, 2005).
2000; 2003; Ball et al., 2001). It is a shared belief that if a teacher does not know mathematics, then he/she cannot teach mathematics (Ball & Bass, 2003; Murphy, 2006). However, one of the first attempts to investigate the relevance between teachers’ knowledge and students’ achievement showed that the number of advanced mathematics courses taken by teachers, was not relevant to students’ achievement (Begle, 1979). Even though Begle’s findings as well as views concerning research in mathematics education, were later doubted by critics (see, e.g., Elerton & Clements, 1998; Howson, 1980), many others since then have argued in favour of this view, i.e. that advanced mathematics courses offered by mathematicians for mathematicians might be of no value to teachers and can even have negative effects on their pedagogical approaches (Cooney & Wiegel, 2003; Davis & Simmt, 2006; Murphy, 2006). Relevant studies by Askew, Brown, Rhodes, Wiliam, and Johnson (1997), Ball and Bass (2000; 2003), Ball et al. (2001) and Davis and Simmt (2006), point out a need for different mathematics for teachers. Davis and Simmt (2006) suggest that teachers’ mathematical knowledge is not a matter of “more of” or “beyond”, but it is of a different quality. This ‘new’ mathematical knowledge should foster teachers’ understanding of mathematics in a broader and unifying sense, so as to enable them to make connections across mathematical domains and help students build a coherent mathematical knowledge (Askew et al., 1997; Ball & Bass, 2000; 2003; Ball et al., 2001). Furthermore, teachers are to anticipate the way mathematical ideas change and grow; hence, they should also be able to make connections across time, as mathematical ideas develop and extend (Ball & Bass, 2003).

Even though there is no general agreement as to the level or the type of mathematical knowledge teachers should have, we see from the above discussion that the multifaceted characteristic of connectedness - across various mathematical domains, conceptual aspects of one same notion, or even across time as a mathematical idea develops - is widely present in the literature. Another example is the study of Ma (1999), in which she compares Chinese and U.S. elementary teachers’ mathematical knowledge. Ma uses the term “profound understanding of fundamental mathematics” to describe teachers’ mathematical knowledge, which is later described by Ball and Bass (2003) as “a kind of connected … and longitudinally coherent knowledge of core mathematical ideas” (p.4). Ball and Bass (2000) explain that “depth” for Ma is the connecting of ideas with the larger and more powerful ideas of the domain, “breadth” is related to the connecting of ideas of similar conceptual power and “thoroughness” is what groups everything together into a coherent whole.

The need for a connected mathematical knowledge for teachers is also suggested by studies that focus on the teaching and learning of specific mathematical ideas. One such example is a study by Lamon (1996) of children’s partitioning strategies and the development of their unitizing process. In her section Implications for Instruction she writes

The many personalities of subconstructs of rational number that children must conceptually coordinate may all be understood as compositions and recompositions of
units. Because the rational numbers are a quotient field, partitioning itself is an operation that plays a role in generating each of those subconstructs ... Students need extensive presymbolic experiences involving these conceptual and graphical mechanisms in order to develop a flexible concept of unit and a firm foundation for quantification, to develop the language and imagery needed for multiplicative reasoning, and to conceptually coordinate the additive and multiplicative aspects of rational numbers. (p.192)

All of the above concerning the multiple faces of a rational number, the multiplicative versus the additive structure, the development of flexible concepts, etc. are characteristics of the students’ desired knowledge. One can only begin to understand how complex, rich, versatile and flexible teachers’ content knowledge should be, so as to be able to correspond to such high and demanding expectations.

WHY ABSTRACT ALGEBRA

Abstract Algebra is an advanced mathematics course usually meant for mathematics students, and it is no surprise that the literature concerning the pedagogy of Abstract Algebra, focuses on the teaching of the course to university students majoring in mathematics, or to high-school mathematics teachers (see, e.g., Burn, 1996; Dubinsky, Dautermann, Leron, & Zazkis, 1994; Leron & Dubinsky, 1995; Simpson & Stehlíková, 2006). The subject is hard enough, even when taught to mathematics majors and we have the, apparently famous, quote: “The teaching of Abstract Algebra is a disaster, and this remains true almost independently of the quality of the lectures” (Leron & Dubinsky, 1995, p. 227; Simpson & Stehlíková, 2006, p.347). So why should one attempt to teach notions of Abstract Algebra to pre-service primary school teachers?

History, Character and Basic Definitions

Until the end of the eighteenth century, Algebra was concerned with mainly the study of polynomial equations and was regarded as a generalization of Arithmetic. The nineteenth century was for Algebra the period of transition, of complete reform. Some of the basic characteristics of the mathematics of the nineteenth century were a turn toward rigor and the need for axiomatization, and the emergence of abstraction. Following Geometry, Algebra became another branch of Mathematics that mathematicians tried to axiomatize. Furthermore, the attention was now turned to the study of mathematical objects, such as vectors, matrices, transformations, etc, and various operations acting on them, which expanded the role of Algebra to the study of form and structure, giving birth to Abstract Algebra. It is in this century that we have the explicit formulation of the fundamental concepts of a group, ring and field, which grew out of the work of many mathematicians such as Galois (1811-1832), Cayley (1821-1895) and Dedekind (1831-1916). These three concepts constitute the focal point of the Abstract Algebra course that we propose, since they have by definition a unifying nature. In the preface of his book, Herstein (1999) mentions that one aspect of the role of Abstract Algebra is “… that of a unifying link between disparate parts of mathematics …” (Herstein, p.xi); and Robert (1987) refers to concepts of the
theory of groups as “unifying and generalizing concepts” (in Dorier, 1995, p.175). But what is it exactly that gives the concept of a group such unifying powers? Let us first recall the definition:

Definition 1: A non-empty set \( G \) is said to be a group if we define an operation \( * \) in \( G \) such that: (1) If \( a, b \in G \) then \( a*b \in G \), and we say that \( G \) is closed under \( * \), (2) Given \( a, b, c \in G \) then \( a*(b*c) = (a*b)*c \), and we say that the associative law holds in \( G \), (3) There exists a special element \( e \in G \) such that \( \forall a \in G, a*e = e*a = a \). This element \( e \) is called the identity or unit element of \( G \), (4) \( \forall a \in G \) there exists an element \( b \in G \) such that \( a*b = b*a = e \). We write this element \( b \) as \( a^{-1} \) and we call it the inverse of \( a \) in \( G \). If in addition, we have that (5) \( \forall a, b \in G, a*b = b*a \), i.e. the commutative law holds, then we say that \( G \) is an abelian group.

From the definition alone, one can recognize the shift of attention from specific objects and operations, to the interrelationships between objects produced under the action of some operation. Seeing for example that \((\mathbb{Z}, +)\) and \((\mathbb{Q}^*, \cdot)\) are both groups, implies, among other things, the realization that \(0\) in the first example and \(1\) in the second play exactly the same role, and therefore the ‘identification’ of these two seemingly unrelated objects. Similarly, \(z\) and \(-z\) connect together in the same way that \(q\) and \(q^{-1}\) in the second example do, where \(z\) is any integer and \(q\) any rational.

Another very useful example involves the even and odd numbers and the realization that \((2\mathbb{Z}, +)\) forms a group whereas \((2\mathbb{Z}+1, +)\) does not since closure fails to hold in the second case. Proving this requires seeing the even and odd numbers in their general form as \(2k\) and \(2k+1\) respectively, where \(k\) any integer. And this in turn, requires the learning as well as the use of the definitions of these two kinds of numbers, which is a type of knowledge that is expected from the teacher (see Ball & Bass, 2000, the Introduction).

Definition 2: A non-empty set \( R \) is said to be a ring if there are two operations \( + \) and \( \cdot \) such that: (1) \((R, +)\) is an abelian group, (2) if \( a, b \in R \) then \( a*b \in R \), (3) \( a*(b*c) = (a*b)*c \), for \( a, b, c \in R \), (4) (i) \( a*(b+c) = a*b + a*c \) and (ii) \( (b+c)*a = b*a + c*a \). If in addition we have that (5) \( \forall a, b \in R, a*b = b*a \), then we say that \( R \) is a commutative ring and the axiom 4(ii) is unnecessary.

The axioms for a ring look familiar since they are a generalization of what happens to the integers. The object \((\mathbb{Z}, +, \cdot)\) is indeed a commutative ring with unit. One can see that what rings are ‘missing’ from behaving like the rationals or the reals are the multiplicative inverses and of course the need for the multiplicative identity element, equivalently called unit, which we denote by \(1\). What we are about to define as a field, has exactly these two extra axioms. Therefore

Definition 3: A non-empty set \( F \) is said to be a field if there are two operations \( + \) and \( \cdot \) such that: (1) \((F, +, \cdot)\) is a commutative ring with unit, (2) for every non-zero \( a \in F \) there is an element \( a^{-1} \in F \) such that \( a*a^{-1} = 1 \).
And so a teacher can now say that \((\mathbb{Q}, +, \cdot)\) is a field. What have we exactly achieved by that? Let us recall the three characteristics of teachers’ mathematical knowledge given by Ma (1999), which we discussed in the first section; namely “depth”, “breadth” and “thoroughness”. We have for example the pair \((q, -q)\) and we connect it to the bigger, more powerful idea of inverse; or we have 1 and we connect it to the more powerful idea of the identity element. These kinds of connections are associated with the notion of “depth”. Furthermore, we connect the seemingly different pairs \((q, -q)\) and \((q, q^{-1})\), by putting them under the category “inverses”; or we connect 1 and 0 by naming them “identity” elements. In other words, we see how \((q, -q)\) and \((q, q^{-1})\), or 0 and 1, are of similar conceptual power. These connections are indications of “breadth”. And finally, by verifying all those axioms for a field, we are able to put everything we know about the rationals together, behind the symbol \((\mathbb{Q}, +, \cdot)\). Hence we have achieved “thoroughness”.

**Groups, Rings, Fields - A Semiotic Perspective**

In this section, I will formalise the discussion and the results from above, by following Peirce’s semiotics. Through Peirce’s semiotic hierarchy I expect to show how the learning of the notions of a group, ring and field can lead to the making of the desired connections that we discussed in the first section, and that these connections are in the highest level of the hierarchy.

A semiotic approach to mathematical activity is an alternative to a psychological approach, which focuses on the mental aspects of learning, and to performance focused perspectives, studying only students’ behaviours. According to Ernest (2006), a semiotic approach transcends the other two above-mentioned approaches, as it “…draws together the individual and social dimensions of mathematical activity…” (p.68). Semiotics is the study of signs, especially as elements of a system, and as such, semiotics serves as a natural theoretical framework for the learning and teaching of mathematics, since mathematics requires certain sign systems “to keep a record of and code the knowledge” (Steinbring, 2006) and mathematization “means representing problems or facts by means of symbols, indices and relational representations” (Hoffmann, 2006, p.279).

For Peirce, a sign is anything that “stands for something (its object)” in such a way as to generate meaning (called its interpretant) (Otte, 2006, pg.23). The signs are of three types. On the first level we have the icons, which are, as the word says, icons – pictures – likenesses of what they represent. A number is an example of an icon; i.e. “1” stands for the number or the idea of “one”. If now we choose to write “m” to imply any integer, then “m” becomes a name, an index, of something existing. An index is on the second level of the hierarchy. Other examples of indices are the ideas of “unit”, “additive inverse” and “identity element”, when these are considered as names or categories for the appropriate icons. One way to think of what an index does is that it organizes icons “in higher order relationships” (Davis & McGowen, 2001, p.10). However, if we make the realizations (i.e. prove) that for every integer
\[ m, -m \text{ is the “additive inverse” of } m, \ 1 \cdot m = m, \ 0 + m = m, \text{ etc.} \] then we are on the third level of the hierarchy, the symbolic level, as the aforesaid realizations are of a law-like nature and they are generalizations regarding the behaviour of the integers. In Peirce’s words, the symbol, which is on the third and highest level of the hierarchy, refers to its object “by virtue of law” (in Radford, 2000, p.252). But we see that it is through an index that we can say that the integers have a certain quality. It is the index “additive inverse” for example, that allows us to say that all integers have an additive inverse. In the words of Peirce again:

A symbol is a conventional sign which being attached to an object signifies that object has certain characters. But a symbol, in itself, is a mere dream; it does not show what it is talking about. It needs to be connected with its object. For that purpose, an index is indispensable. No other kind of sign will answer the purpose (in Otte, 2006, p.30).

If we now consider all the indices that appear in the definition of a ring and the way these organize the icons of the integers, such as the numbers themselves, the signs “=”, “+”, etc., then we have, according to Peirce, the connection between the symbol, if I may denote it by “the ring \((\mathbb{Z}, +, \cdot)\)”, and the integers. And as we attach to the integers the symbol “the ring \((\mathbb{Z}, +, \cdot)\)” we automatically ‘see’ in the integers all those properties that we all learn from our early years until university. Deacon (1977) refers to the shift to the symbolic level of the hierarchy as:

… a way of off-loading redundant details from working memory, by recognizing a higher order regularity in the mess of associations, a trick that can accomplish the same task without having to hold all details in mind. (in Davis & McGowen, 2001, p.10)

The notions of group, ring and field behave exactly like that: they allow which ever object we are investigating, i.e. the integers, the complex numbers, the \(2\times2\) invertible matrices, to be “apprehended, pretty much all at once” (Davis & McGowen, 2001, p.10), i.e. through the symbol “the ring \((\mathbb{Z}, +, \cdot)\)”, “the group \((\mathbb{Z}, +)\)”, “the field \((\mathbb{C}, +, \cdot)\)”, “the group of \(2\times2\) invertible matrices \(GL(2, \mathbb{R})\)”, etc., without having to hold all details in mind. As such, the notions of group, ring and field are systems of symbols and they form a part of a connected symbolic system, the symbolic associations of which are being understood from the “myriad connections” between the indexes (Davis & McGowen, 2001, p.10) organized by this system.

From the above discussion we see that by moving up on the hierarchy, we achieve a shift in understanding: a shift from the specific and isolated, i.e. the number 1, 0, 2, -2, \(\frac{1}{2}\), to ideas such as “unit”, “additive or multiplicative inverse”, “the ring \((\mathbb{Z}, +, \cdot)\)”, etc., which connect together seemingly different parts of our knowledge, putting them into a coherent whole.

**IN-CLASS OBSERVATIONS AND REFLECTIONS**

This last section aims to provide the reader with some concrete examples from, and suggestions, if you will, regarding the teaching of basic notions of Abstract Algebra to pre-service primary school teachers.
While being a Lecturer in Mathematics Education at a university in Cyprus, I taught a course for pre-service primary school teachers, called General Topics in Mathematics, for two consecutive semesters. The spring semester course ran over a period of thirteen weeks and my students and I met once a week, for three hours. The summer semester course was seven weeks long and we met twice a week, for three hours each time. I kept the syllabus for both courses the same. I chose a variety of topics from traditional Algebra, Logic, Set Theory, etc. and during the last three lectures I gave a brief introduction to the complex numbers and the notions of group, ring and field, including examples from areas they had already seen before, as well as some basic examples from modular arithmetic. In class observations and discussions revealed similar reactions and results from both classes. I observed that, even though the students were exposed to these notions only for a very short period of time, the desired results started to happen and the students’ main difficulties were related mainly with the new to them area of modular arithmetic. This is why I believe that a one-semester course can give students enough time to digest this new way of thinking as well as the examples from any new mathematical areas introduced to them.

The following are some examples from in-class observations as well as from the students’ answers in the final exam: (1) The students seemed to have appreciated more the importance of the distributive law, as they saw how it connected together the abelian groups $\langle Q, + \rangle$ and $\langle Q^\times, \cdot \rangle$ for example, in order to give the well-known field $\langle Q, +, \cdot \rangle$. (2) It appeared that the notion of closure was being brought to their attention for the first time. They did not understand the necessity of this axiom in the definition of a group. Their reactions were pretty much along the line “if you add a number you will get a number and that is pretty much obvious, so why do we need this statement?” I asked them: what happens if you add 0 and 4? -2 and 100? 1000 and 44? So we concluded that when you add two even numbers you seem to get an even number. What followed was the realization that this does not happen with the odd numbers. The next step was to prove that even + even = even and odd + odd ≠ odd. (3) Another worth mentioning observation related to the notion of closure again, appeared when I emphasized in the definition of a group the need for the inverse $a^{-1}$ to also be a member of the set $G$. Once again, they did not really see the need for this until I mentioned the example of the multiplicative inverse of 3; they all said it is 1/3 but when I asked what if $G = \mathbb{Z}$, they realised that since 1/3 is not an integer then $(\mathbb{Z}^\times, \cdot)$ cannot be a group. (4) One of the earlier topics that were presented in this course was the notion of function. The concept of a function came up again in one of the examples, were they had to prove that the following set $G$ of functions is a group: $G = \{T_{a,b} : R \rightarrow R \mid T_{a,b}(r) = ar + b, a, b \in R, a \neq 0\}$. At first, there was an immense reaction (expected) from the students about the impossibility of this exercise due to its highly symbolic character, or in their words, to the appearance of “too many letters”. When I explained that the function $T_{a,b}$ is merely a generalization of functions like the ones they saw before, for example like the function $f(x) = 3x+2$, etc., then the complaints seized. It was very interesting and rewarding to see their
surprised faces when they realized that (a) the identity element is the function $T_{1,0}$ and that this object behaves in exactly the same manner as 0 in $(\mathbb{Z}, +)$ for example, or 1 in $(\mathbb{Q}^*, \cdot)$, (b) even if this problem is so different from everything they saw before, the procedure to find the inverse element of $G$ was exactly the same as the steps they followed earlier in the course, in order to find the inverse of a specific function. In other words, they were very surprised to see that the same method could work in such different, for them, situations. They presented difficulty however, which was also shown in the final exam, in showing that the inverse $T_{a,b}^{-1}$ is still a member of $G$; i.e. even though almost everyone showed that $T_{a,b}^{-1}(r) = \frac{1}{a}r - \frac{b}{a}$, not everyone could see why we are not done until we write $T_{a,b}^{-1} = T_{1/a, -b/a}$, $1/a \neq 0$. (5) Difficulty for students also presented in the case of modular arithmetic. Since there was not enough time to really digest this new kind of “numbers”, even though they were pretty quickly able to find the additive and multiplicative inverses of $\mathbb{Z}_5$ for example and they appeared to understand why not every non-zero element of $\mathbb{Z}_6$ has a multiplicative inverse, in the exam when they were asked to explain why $(\mathbb{Z}_6^*, \cdot)$ is not a group, the majority did not succeed. However the majority succeeded in the two more familiar examples where they had to explain why $(\mathbb{Z}_6^*, +)$ and $(\mathbb{Z}_6^*, \cdot)$ are not groups. (6) The two new topics, modular arithmetic and complex numbers, created some problems for the students because of time shortage. However, the introduction to complex numbers that the students had, gave them some sort of continuity regarding the number system, it helped them realize and understand the, up until then unknown to them, relationship $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$, and triggered their curiosity – they asked if and where these sets of numbers stop, if there are other number systems, some recalled the $(i, j, k)$-triples they had learned at some point in high school and asked if these were related to the complex numbers, and many more. Modular arithmetic also triggered their curiosity. Even if the introduction to the topic was short, the students were still able to appreciate the mathematical innovation and imagination while ‘discovering’ new kinds of inverses such as 5, being the multiplicative inverse of itself mod(6), or that 2 does not have an inverse mod(6). Even more interesting for them was when they saw how in some systems the product of two non-zero elements can be zero, for example $2 \cdot 3 \equiv 0$ mod(6).

All of the above different examples from the area of functions, modular arithmetic, as well as the various number systems, were solved by using the same ‘tools’, i.e. the concept of group, ring and field. As is also pointed out by Dorier (1995, p.1), concepts such as the concept of a group, were invented not only to solve new problems but “mainly to find general methods to solve different problems with the same tools” (Dorier, 1995, p.1). This ‘proves’ in a way how the unifying character of these ‘tools’ can be used in the training of future teachers, in order to help them achieve the desired connections described in the previous sections.

REFERENCES


Despite the common myth that teaching is little more than common sense or that some people are just born teachers, effective teaching practice can be learned. First, what does it take to be proficient at mathematics teaching? If their students are to develop mathematical proficiency, teachers must have a clear vision of the goals of instruction and what proficiency means for the specific mathematical content they are teaching. They need to know the mathematics they teach as well as the horizons of that mathematics—where it can lead and where their students are headed with it. Implications for teaching algebra To be effective a teacher has to be aware of pupils' individual approaches as well as orchestrate learning so that pupils develop knowledge of mathematics that is recognized by communities outside school. Implications for teacher education Teachers need support and guidance in order to recognize the essential nature of algebraic activity. Implications for national curricula 5.1 Nature of school algebra 5.2 Algebra as a language 5.3 Changing emphasis in school algebra 5.4 Implications for the timing of algebra teaching 5.5 Implications for teaching algebra 5.6 Implications for assessment 5.7 Implications for the development of curriculum materials 5.8 Implications regarding new technologies 5.10 Implications. I rst taught an abstract algebra course in 1968, using Herstein's Topics in Algebra. It's hard to improve on his book; the subject may have become broader, with applications to computing and other areas, but Topics contains the core of any course. Unfortunately, the subject hasn't become any easier, so students meeting abstract algebra still struggle to learn the new concepts, especially since they are probably still learning how to write their own proofs. of a function of x if and only if no vertical line intersects the curve more than once. Explain why this agrees with Definition 2.1.1. 22. The "Horizontal Line Test" from calculus says that a function is one-to-one if and only if no horizontal line intersects its graph more than once. Explain why this agrees with Definition 2.1.4. Why abstract algebra for pre-service primary school teachers. Article. Full-text available. Cognitive Acceleration programs have been successful in promoting reasoning skills in school students and in changing the pedagogy of the in-service teachers applying them. The novelty of this study is that it implemented, for the first time, that socio-constructivist approach with prospective teachers. One is the discourse generated by pre-service teachers when adopt the role of students of any level who have to solve a task proposed in the classroom. The other discourse is linked to the adoption of a role close to their future professional work. From a commognitive approach, this article focuses on the discourse generated by pre-service primary teachers who are solving didactic-mathematical tasks. Our aims are to study the characteristics of the aforementioned discourse and, through these characteristics, identify whether a discourse close to the one of primary teachers is beginning to emerge. The sources of data were audio-recordings of group discussions and group reports. Two different discourses were identified in our results.